$$=[GEF][FEG][GFE]E\ldots.(57)=[GEF][EFG][GEF]E\ldots.(41)$$

$$=[EFG]E,$$

since $[GEF][GEF]$, as in (57), equals $+1$.

68. *If A, B, C are simple quantities the sum of whose orders equals n, the order of the space of these quantities.*

$$[AB.AC]=[ABC]A.$$

The proof of this formula (based on (67) on account of its length, is omitted, as are also those of the next four formulas given below.

69.—71. *If A, B, C are simple quantities whose product is of the 0th order.*

69. $$[AB.AC]=[ABC]A.$$

70. $$[AB.BC]=[ABC]B.$$

71. $$[AC.BC]=[ABC]C.$$

72. *If A, B, C are simple quantities and the sum of the orders of A and C equals the order of the space considered and B is subordinate* (18) *to A, then*

$$[A.BC]=[AC]B \text{ and } [CB.A]=[CA]B.$$

Remark.—It seems proper to state here that the matter contained in Chapters II—V is taken direct from Grassmann's Ausdehnungslehre of 1862. What the writer has done has been to cut out everything which was not essential to the development of the main principles of the work. What to insert and what to omit constitutes the chief difficulty. In the following chapters (except Chapter VIII) we shall not follow Grassmann very closely.

MATHEMATICAL INDUCTION.

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There is such a general lack of presentation of the governing principles of mathematical induction in the text books which refer to the subject, and failure to give a working rule for its application, that it is thought a presentation of such a working rule specifically stated and not left to inference from examples merely, would be acceptable.

Mathematical Induction (not entering into the question of the appropriateness of the name) is a method of proof used where a primary operation and a

secondary law have produced the same series of results to see if the supposed secondary law is valid.

It may be conveniently divided into two cases :

I. When the primary operation is an algebraic one and the secondary law a functional one.

II. When the primary oporation is a functional one and the secondary law an algebraic one

The necessity for a proof that the secondary law is valid is seen when we find that many apparent secondary laws which hold for a few terms fail later on, in other words, that because an apparent law holds for ten, twenty or thirty terms of a series, we are not at all justified in inferring that it will hold throughout.

A few examples will illustrate this :

x^2+x+17 is prime for sixteen values of x, beginning with $x=0$, but the law fails for the seventeenth term of the series.

x^2+x+41 is likewise prime for forty terms, but the law fails on the forty-first.

The sum of the divisors of $n!$ is $\frac{1}{2}(n+1)!$ for a few terms, after which the law fails. And so on for many apparent secondary laws.

I. When the primary operation is a simple algebraic one, and the supposed secondary law is a more complex or functional one.

In this case Mathematical Induction consists in applying the primary operation to the general result produced by the secondary law to see if the next term of the series thus produced by the primary operation is the same as would result from an application of the secondary law. If it is, and the operand was correct, the secondary law is correct.

Example.—$1^3+2^3=9(2-1)^2$, $1^3+2^3+3^3=9(3-1)^2$, giving the apparent secondary law $1^3+2^3+\ \ldots.\ n^3=9(n-1)^2$.

To test this, add another term (primary operation) giving

$$1^3+2^3+\ldots\ldots(n+1)^3=9(n-1)^2+(n+1)^3\pm 9n^2,$$

and therefore the secondary law is not correct.

Example.—$1^3+2^3=\left(2\frac{2+1}{2}\right)^2$, $1^3+2^3+3^3=\left(3\frac{3+1}{2}\right)^2$, giving the apparent secondary law,

$$1^3+2^3+\ldots\ldots n^3=\left(n\frac{n+1}{2}\right)^2.$$

Apply the primary operation by adding another term, giving

$$1^3+2^3+\ldots\ldots(n+1)^3=\left(n\frac{n+1}{2}\right)^2+(n+1)^3=\left[\ (n+1)\ \frac{n+2}{2}\right]^2$$

and the secondary law is correct.

II. When the primary operation is a functional one and the secondary law is a simple algebraic one.

In this case mathematical induction consists in stating the primary functional operation for two general terms in succeession, and taking any convenient multiple of the first from a convenient multiple of the second. If this difference conforms to the supposed secondary law, the secondary law is true provided the first functional result is true.

Thus let

$$f(x)=F\phi$$

be a term of the series, where x is the element whose change produces the term of the series, and ϕ is the expression which enters as an element of the secondary law. If now

$$A.f(x+1)-B.f(x)=F_1\phi$$

where F_1 has the same algebraic form as F, except as to the values of the constants, then

$$f(x+1)=F_2\phi$$

where F_2 has the same algebraic form as F and F_1 except as to the value of the constants. Hence the form of the function F being persistent for two terms, the law is true if true for F.

A special case is where $F\phi=C.\phi$, C being a mere factor.

Example. — $x^3-x=\binom{6\alpha}{24\beta}$ according as x is $\binom{\text{even}}{\text{odd}}$, α and β being factors, holds for some cases.

1°. $x=2n+1$.

$f(x)=(2n+1)^3-(2n+1)=8n^3+12n^2+4n$.

$f(x+1)=(2n+3)^3-(2n+3)=8n^3+36n^2+52n+24$.

$f(x+1)-f(x)=24n^2+48n+24=24\beta$. Q. E. I.

2°. $x=2n$.

$f(x)=(2n)^3-2n=8n^3-2n$.

$f(x+1)=(2n+2)^3-(2n+2)=8n^3+24n^2+22n+6$.

$f(x+1)-f(x)=24n^2+24n+6=6\alpha$. Q. E. I.

Example.—$f(x)=5^{2x+2}-24x-25=576\alpha$ holds for $x=1, 2, 3$.

$f(x+1)=5^{2x+4}-24(x+1)-25$.

$f(x+1)-25f(x)=25(24x+25)-24x-49=576(x+1)=576\alpha$. Q. E. I.